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CALCULUS OF VARIATIONS: FURTHER DISCUSSION OF THE FIRST VARIATION AND THE SOLUTION OF THE DIFFERENTIAL EQUATION $G = 0$ FOR SEVERAL INTERESTING EXAMPLES.

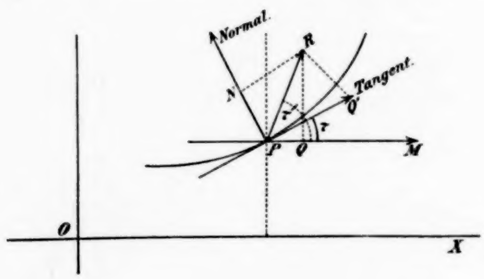
By Dr. Harris Hancock, Chicago, Illinois.

We saw ("The Annals," Vol. XI, p. 28) that

$$\delta I = \int_{t_0}^{t_1} G\,(y'\xi - x'\eta)\,dt + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1}, \tag{1}$$

and we wish now to interpret $G\,(y'\xi - x'\eta)\,dt$.

Consider any point P of the curve, and suppose it is subjected to a sliding



in the direction PR; also suppose that ds has the same sign as dt, so that when we write

$$PR = v\,,$$

$$PQ' = v_1\,,$$

$$PW = w\,,$$

$$\angle\; RPM = \tau'\,,$$

$$\angle\; QPM = \tau\,,$$

then is

$$x' = \frac{dx}{dt} = \frac{dx}{ds}\cdot\frac{ds}{dt} = \frac{ds}{dt}\cos\tau$$

and

$$y' = \frac{dy}{dt} = \frac{dy}{ds}\cdot\frac{ds}{dt} = \frac{ds}{dt}\sin\tau\,.$$

Substitute these values in (1) and we have :

$$\delta I = \int_{t_0}^{t_1} G\,(\xi \sin\tau - \eta \cos\tau)\, ds + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1}. \qquad (2)$$

To denote that a sliding takes place along the line PR we must make (see p. 25) the substitution :

$$\begin{array}{c|l} x & x + e\xi\,, \\ y & y + e\eta\,, \end{array}$$

where e is the variable parameter.

Denote the whole sliding that takes place along PR by v, then the component in the x direction, that is ξ, is

$$\xi = v \cos\tau'\,,$$

and

$$\eta = v \sin\tau'\,.$$

Further decomposing v into two components in the direction of the tangent and the normal to the curve at P, and denoting these components respectively by v_1 and w, we have

$$v_1 = v\cos(\tau' - \tau)\,,$$

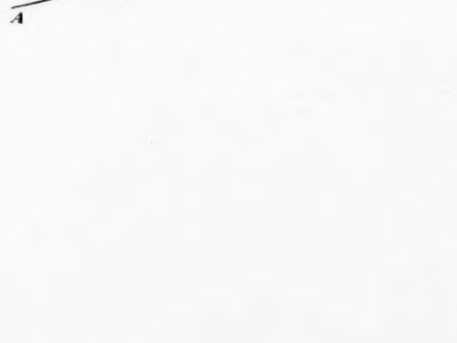
$$w = v\sin(\tau' - \tau) = v\sin\tau'\cos\tau - v\cos\tau'\sin\tau$$

$$= \eta\cos\tau - \xi\sin\tau\,.$$

Hence from (2)

$$\delta I = -\int_{t_0}^{t_1} Gw\,.\,ds + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\cdot\ \right]_{t_0}^{t_1},$$

from which is seen *that only that component of the sliding enters under the sign of integration which is in the direction of the normal.*



It is shown below that all slidings in the direction of the tangent can give only such terms that are completely integrable.

When a curve slides along into itself, this means nothing else than that we traverse the curve by referring the points in another manner to an independent variable. A sliding of the curve into itself is clearly nothing else than a sliding of the single points in the directions of their tangents; and we will now show that this sliding gives only such terms of the first variation which are free from the sign of integration. If we make this slide, then the integral I has either in all the elements the same quantities, or it happens that at the beginning a portion of the curve is cut off and at the end a portion enters in addition, so that at the beginning elements are lost, and at the end additional elements enter.

By means of the formulæ below we will prove in a direct manner that *the variation in the direction of the tangent brings forth only such terms for the first variation that are free from the sign of integration.*

Writing (see "The Annals," p. 26)

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right] dt,$$

and noting that $\tau = \tau'$ when the sliding is in the direction of the tangent, we have

$$\xi = v \frac{dx}{ds} = v \cos \tau',$$

and

$$\eta = v \frac{dy}{ds} = v \sin \tau',$$

and hence

$$\delta I = \int_{t_0}^{t_1} \left\{ v \left[\frac{\partial F}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial F}{\partial y} \cdot \frac{dy}{ds} \right] + \frac{\partial F}{\partial x'} \cdot \frac{d}{dt} \left[v \frac{dx}{ds} \right] + \frac{\partial F}{\partial y'} \cdot \frac{d}{dt} \left[v \frac{dy}{ds} \right] \right\} dt.$$

But

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} \cdot \frac{d}{dt} \left[v \frac{dx}{ds} \right] = \left[\frac{\partial F}{\partial x'} \cdot v \frac{dx}{ds} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} v \frac{dx}{ds} \cdot \frac{d}{dt} \left[\frac{\partial F}{\partial x'} \right] dt;$$

so that

$$\delta I = \int_{t_0}^{t_1} \left\{ v \frac{dx}{ds} \left[\frac{\partial F}{\partial x} - \frac{d}{dt} \left[\frac{\partial F}{\partial x'} \right] \right] + v \frac{dy}{ds} \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left[\frac{\partial F}{\partial y'} \right] \right] \right\} dt + \left[\frac{\partial F}{\partial x'} \cdot v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} G (y'\xi - x'\eta)\, dt + \left[\frac{\partial F}{\partial x'} v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1}.$$

But

$$(y'\xi - x'\eta) = \left[y'v\frac{dx}{ds} - x'v\frac{dy}{ds}\right] = \frac{v}{ds}\left[y'\frac{dx}{dt} - x'\frac{dy}{dt}\right]dt = 0\,,$$

so that everything under the sign of integration drops out, leaving

$$\delta I = \left[\frac{\partial F}{\partial x'}v\frac{dx}{ds} + \frac{\partial F}{\partial y'}v\frac{dy}{ds}\right]_{t_0}^{t_1}.$$

Hence if we have any sliding, and resolve this sliding into two components, of which the one is parallel to the direction of the tangent, and the other is parallel to the direction of the normal, then the result of the sliding in the direction of the tangent is seen only in the terms which have reference to the limits, and all these terms are complete differentials under the sign of integration; whereas the sliding in the direction of the normal is seen only under the sign of integration in the first variation.

Note.—This can be generalized to the case where we deal with surfaces; i. e., where we have to do with double integrals.

Integration of the Differential Equation $G = 0$ for the First Three Problems.*

Problem I. Problem of the surface of rotation which is to have the least surface area.

We have already derived the integral (see "The Annals," Vol. IX, p. 183)

$$S = 2\pi\int_{t_1}^{t_1} y\sqrt{x'^2 + y'^2}\,dt\,;$$

so that here

$$F = y\sqrt{x'^2 + y'^2}\,;$$

and consequently

$$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}},\quad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{x'^2 + y'^2}}.$$

From this follows, that

$$\frac{\partial F}{\partial x'}\ \text{and}\ \frac{\partial F}{\partial y'}$$

are proportional to the direction cosines of the tangent at any point $x(t)$, $y(t)$; and since, as shown above, Vol. XI, p. 31, these quantities vary everywhere in a continuous manner, it follows also that the direction of the curve varies

* These problems were stated by Prof. Weierstrass in his lectures (cf. the "Annals," Vol. IX, p. 179), and the solutions that follow are essentially due to him.

everywhere in a continuous manner except for the case where y can be $= 0$.

In this last case the curve consists of several regular traces; *in general*, however, of only one.

Since x is not contained explicitly in F, it is expedient to use $G_1 = 0$ instead of $G = 0$.

The differential equation for the curve that we are seeking is then

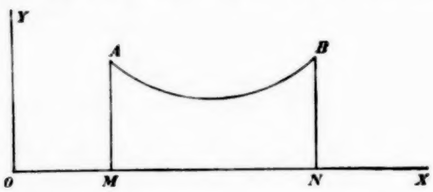
$$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}} = C,$$

where C is an arbitrary constant.

If we take the arc s as independent variable instead of the variable t, the above equation may be written:

$$y\frac{dx}{ds} = C.$$

Suppose that $C = 0$, then C must retain this value also in the whole interval $t_0 \ldots t_1$, that is, along the curve AB; and since y is not zero for the point A, it follows that $\frac{dx}{ds} = \cos a = 0$ (where a is the angle, which the tangent makes with the x-axis), $\cos a$ must remain $= 0$ until $y = 0$; that is, the point which



describes the curve must move upon the ordinate A to the point M. At this point $\frac{dx}{ds}$ cannot and must not equal to zero if the point is to move to B. Hence at M there is a sudden change in the direction of the curve. The point then moves farther upon the x-axis, but must again leave this axis in order to go to B, which is only possible (when $C = 0$) by having $\frac{dx}{ds} = 0$, and then the point moves on the ordinate of B. Hence for the case $C = 0$, the curve, which satisfies the differential equation, consists of the broken line $AMNB$.

But if $C \neq 0$, then $C \neq 0$ throughout the whole extent of the curve. Since $y \neq 0$ at the beginning, it follows that $\frac{dx}{ds}$ is finite and $\neq 0$ and varies throughout the whole extent of the curve in a continuous manner.

Hence y must also vary everywhere in a continuous manner and can

never be zero, that is, the curve consists of one single regular trace, which lies wholly on the positive side of the x-axis.

From the relation

$$y\frac{dx}{ds} = C$$

we have

$$y^2 dx^2 = C^2(dx^2 + dy^2)$$

a differential equation which has been treated at length in "The Annals," Vol. X, p. 84 and p. 159.

Problem II. Problem of the brachistocrone (cf. "The Annals," Vol. IX, p. 183).

The time of falling of the movable point expressed in terms of $x(t)$ and $y(t)$ from the point A to the point B is

$$T = \int_{t_0}^{t_1} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + \alpha^2}}\, dt\,.$$

In order that this integral should in reality express the time of falling, since within the integral the time and therefore also the increment dt is an essentially positive quantity, the two roots must always have the same sign. And since $\sqrt{4gy + \alpha^2}$ can always be chosen positive, it follows that $\sqrt{x'^2 + y'^2}$ must within the interval $t_0 \ldots t_1$ be positive.

It might happen, however, if we express x and y in terms of t, that x' and y' might both vanish for a value of t within the interval $t_0 \ldots t_1$. In this case the curve has at the point x, y, which belongs to this value of t, a singular point, at which the velocity of the moving point is zero.

Suppose that this is the case for $t = t'$ and that the corresponding point is x_0, y_0, so that we have

$$x = x_0 + A(t - t')^m + \ldots,$$
$$y = y_0 + B(t - t')^m + \ldots,$$

where $m \geqq 2$, and at least one of the two quantities A and $B \neq 0$.

Then is:

$$x'^2 + y'^2 = m^2(A^2 + B^2)(t - t')^{2(m-1)} + \ldots,$$
$$\sqrt{x_1'^2 + y'^2} = m\sqrt{A^2 + B^2}\,(t - t')^{m-1} + \ldots.$$

And here we may suppose $\sqrt{A^2 + B^2}$ positive.

If now m is odd, then for small values of $t - t'$, the expression on the right is positive, and hence $\sqrt{x'^2 + y'^2}$ has always the positive sign.

If on the contrary m is even, $=$ say 2, then in this case the curve has at the point x_0, y_0 a cusp, since here $\sqrt{x'^2 + y'^2}$ has a positive or a negative value according as $t > t'$ or $t < t'$.

If therefore the above integral is to express the time, $\sqrt{x'^2 + y'^2}$ cannot always be put equal to the same series of t, but must after passing the *cusp* be put equal to the opposite value of the series. We therefore limit ourselves to the consideration of a portion of the curve which is free from singular points.

Such limitations must be often made in problems, since otherwise the integrals have no definite meaning.

Hence with this supposition $\sqrt{x'^2 + y'^2}$ will never $= 0$.

We may then write:

$$F = \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}}, \tag{2}$$

and consequently:

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{x'}{\sqrt{x'^2 + y'^2}},$$

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

And from this we conclude in a similar manner as in the first example, that $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$, are proportional to the direction cosines of the tangent of the curve at the point x, y. Since now $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$ vary in a continuous manner along the the whole curve, and since further $\sqrt{4gy + a^2}$ has a definite value which is different from zero, it follows also that the direction of the required curve varies in a continuous manner, or the curve must consist of one single trace.

Also here F is independent of x, and consequently we employ the differential equation $G_1 = 0$; from which we have

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{x'}{\sqrt{x'^2 + y'^2}} = c, \tag{3}$$

where c is an arbitrary constant.

If c was equal to 0, then in the whole extent of the curve c must $= 0$; and consequently, since $\sqrt{4gy + a^2}$ is neither 0 nor ∞, $\frac{x'}{\sqrt{x'^2 + y'^2}} = \cos a$ must always $= 0$, that is, the curve must be a vertical line. And neglecting this self-evident case, c must have a definite value which is always the same for the whole curve and different from zero.

From (3) follows that:

$$dx^2 = c^2 (4gy + a^2)(dx^2 + dy^2).$$

Or, if we absorb $4g$ in the arbitrary constant and write

$$\frac{a^2}{4g} = a\,,$$

we have

$$dx^2 = c^2 (y + a)(dx^2 + dy^2);$$

whence

$$dx = \frac{c(y + a)\,dy}{\sqrt{(y + a)(1 - c^2(y + a))}}.$$

In order to perform this last integration write

$$du = \frac{c\,dy}{\sqrt{(y + a)(1 - c^2(y + a))}}; \tag{5}$$

therefore

$$dx = (y + a)\,du.$$

In the expression for du, write

$$2c^2 (y + a) = 1 - \xi.$$

Then is

$$2(1 - c^2(y + a)) = 1 + \xi,$$

and

$$2c^2 dy = -\,d\xi.$$

Therefore

$$du = -\frac{d\xi}{\sqrt{1 - \xi^2}},$$

and hence

$$\xi = \cos u.$$

Here the constant of integration may be omitted, since u itself is fully arbitrary.

Hence

$$\left.\begin{aligned} y + a &= \frac{1}{2c^2}(1 - \cos u), \\ x + x_0 &= \frac{1}{2c^2}(u - \sin u); \end{aligned}\right\} \tag{8}$$

and from (6)

and these equations represent a cycloid.

The constants of integration x_0, c are determined from the condition that the curve is to go through the two points A and B. Now develop x and y in

powers of u, then in y the lowest power is u^2, and in x it is u^3. So that the curve has in reality a cusp for $u = 0$, and this is repeated for $u = 2\pi, 4\pi, \ldots$.

A and B must lie within such an interval of u (for example within the interval $2n\pi$ and $2(n+1)\pi$), that is, between two consecutive cusps.

The curve may be constructed if we draw a horizontal line through the point $-x_0, -a$, and construct on the under side of this line a circle with radius $\frac{1}{2c^2}$, which touches the horizontal line at the point $-x_0, -a$. Let this circle roll in the positive x direction on the horizontal lines, then the original point of contact describes a cycloid which goes through A and B and which satisfies the differential equation.

Problem III. Problem of the shortest line on a surface.

This problem cannot in general be solved, since the variables in the differential equation cannot be separated, and consequently the integration of this equation cannot be performed.

Only in a few cases has one succeeded in carrying through this integration, and in this manner of representing the curve which satisfies the differential equation through closed expressions.

This is for example the case with the plane, the sphere, and with all other surfaces of the second degree.

As a simple example we will take the problem of the shortest line between two points on the surface of a sphere. The radius of the sphere is put $= 1$, and the equation of the sphere is given in the form:

$$x^2 + y^2 + z^2 = 1 .$$

Now writing:

$$\left.\begin{aligned} x &= \cos u , \\ y &= \sin u \cos v , \\ z &= \sin u \sin v ; \end{aligned}\right\} \tag{1}$$

then $u =$ const. and $v =$ const. are the equations of the parallel circles and of the meridians respectively.

The element of arc is

$$ds = \sqrt{du^2 + \sin^2 u dv^2} ; \tag{2}$$

and consequently the integral which is to be made a minimum is:

$$L = \int_{t_0}^{t_1} \sqrt{u'^2 + v'^2 \sin^2 u} \, dt ; \tag{3}$$

so that here we have

$$F = \sqrt{u'^2 + v'^2 \sin^2 u}\,, \tag{4}$$

and

$$\left.\begin{aligned} \frac{\partial F}{\partial u'} &= \frac{u'}{\sqrt{u'^2 + v'^2 \sin^2 u}}\,, \\ \frac{\partial F}{\partial v'} &= \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}}\,, \end{aligned}\right\} \tag{5}$$

Since F does not contain the quantity v, we will use the equation $G_1 = 0$ and have:

$$\frac{\partial F}{\partial v'} = \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}} = c\,,$$

where c is an arbitrary constant which has the same value along the whole curve.

If for the initial point A of the curve $u \neq 0$, and consequently, therefore, not the north pole of the sphere, then c will only then be everywhere $= 0$, if $v' = 0$, and when, therefore, $v =$ const. And then A and B must lie on the same meridian, and this is a solution of the problem.

If now this is not the case, then always $c \neq 0$, and as is easy to see $c < 1$; we can therefore write $\sin c$ instead of c, and have:

$$\frac{v' \sin^2 u}{\sqrt{u'^2 + \sin^2 u v'^2}} = \sin c\,,$$

or

$$dv = \frac{\sin c\, du}{\sin u \sqrt{\sin^2 u - \sin^2 c}}\,. \tag{7}$$

If we write

$$\cos u = \cos c \cos w\,, \tag{8}$$

then is

$$dv = \frac{\sin c\, dw}{1 - \cos^2 c \cos^2 w}\,;$$

and when integrated this gives:

$$tg\,(v - \beta) = \frac{1}{\sin c}\, tg w\,, \tag{9}$$

where β represents an arbitrary constant.

Eliminating w by means of (8), we have:

$$tg u \cos (v - \beta) = tg c\,. \tag{10}$$

This is the equation of the curve which we are seeking expressed in the spherical coordinates u, v.

In order to study their meaning closer, we may express u, v separately through the arc s, where s is measured from the intersection of the zero meridian with the shortest line.

Through (7) the expression (2) goes into:

$$\partial s = \frac{\sin u \, du}{\sqrt{\sin^2 u - \sin^2 c}},$$

and this owing to the substitution (8) becomes

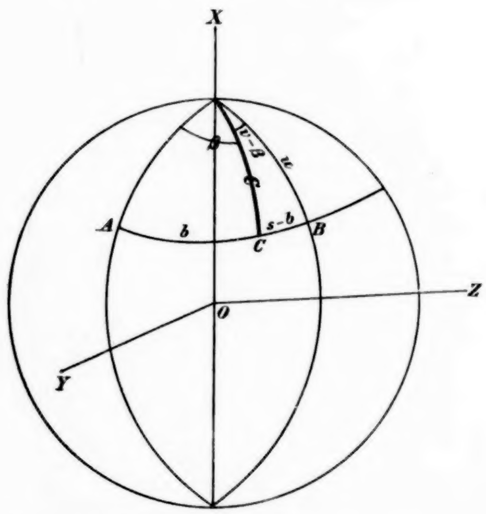
$$\partial s = \partial w,$$

and, therefore, if b is a new constant,

$$s - b = w. \tag{11}$$

Hence from equations (8) and (9) we have the following equations:

$$\left.\begin{aligned} \cos u &= \cos c \cos (s - b), \\ ctg (v - \beta) &= \sin c \; ctg (s - b). \end{aligned}\right\} \tag{12}$$



But these are relations which exist between the sides and the angles of a right angle spherical triangle.

If we consider that meridian drawn from the north pole, which cuts at right angles the curve we are seeking, then this meridian forms with the curve and any other meridian a triangle, to which the above relations may be applied.

Therefore the curve which satisfies the differential equation must itself

be the arc of a great circle. The constants of integration c, b, β are determined from the conditions that the curve is to pass through the two points A and B.

The geometrical interpretation is that c is the length of the geodetic normal from the point $u = 0$ to the shortest line; $s - b$, the arc from the foot of this normal to any point of the curve, and $v - \beta$ the angle opposite this arc, that is, the difference of length between the end points of this arc.

If we therefore assume that the zero meridian passes through A, and if, accordingly, we measure the arc from A, then b is the length of arc of the shortest line from A to the normal, and β the geographical longitude of the foot of this normal.

NOTES ON SOME POINTS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.

By Prof. Maxime Bôcher, Cambridge, Mass.

In the following pages I have tried to treat some points in the theory of linear differential equations in a simpler manner than is ordinarily done, and to insist upon some matters which are usually passed over in silence. In doing this I have not hesitated to confine myself to equations of the second order:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0\,, \tag{1}$$

as everything which is here given can be immediately extended by the same method to equations of higher order. Much of what follows was given at the Buffalo Colloquium held under the auspices of the American Mathematical Society in September, 1896 (cf. Bull. Amer. Math. Soc., Nov., 1896).

§ 1. *Equations with Analytic Coefficients. Non-Singular Points.*

We will begin by considering the case in which the coefficients p and q of the differential equation (1) are analytic functions of the complex variable x. By a non-singular point of such an equation is meant a point at which $p(x)$ and $q(x)$ are analytic,* i. e. about which they can be developed in the form:

$$p(x) = p_0 + p_1(x-a) + p_2(x-a)^2 + \ldots,$$
$$q(x) = q_0 + q_1(x-a) + q_2(x-a)^2 + \ldots.$$

We will suppose that each of these series converges when $|x-a| < R$. We wish to prove the following fundamental Existence Theorem:

There exists a solution of the differential equation (1) *of the form:*

$$y = g_0 + g_1(x-a) + g_2(x-a)^2 + \ldots \tag{2}$$

where g_0 and g_1 are arbitrary constants and where the series converges when $|x-a| < R$.

If we let:

$$f(x, \rho) = \rho(\rho-1) + \rho(x-a)\,p(x) + (x-a)^2\,q(x)\,,$$

* It will be seen that I speak here and in the following sections of a function as analytic at a point if it can be developed in a power series about this point. It will then be analytic throughout a region if it is analytic at each point of the region. The ordinary usage is to restrict the word *analytic* to regions and to use the word *regular* for points. One word seems quite sufficient here.

the result of substituting (2) in (1) is:

$$\sum_{\nu=0}^{\nu=\infty} g_\nu f(x, \nu)(x-a)^{\nu-2} = 0 . \tag{3}$$

Now we have:

$$f(x, \rho) = \sum_{\mu=0}^{\mu=\infty} f_\mu(\rho)(x-a)^\mu ,$$

where $f_0(\rho) = \rho(\rho-1)$, $f_1(\rho) = \rho p_0$, and when $\mu > 1$ $f_\mu(\rho) = \rho p_{\mu-1} + q_{\mu-2}$. Let us substitute this development of $f(x, \rho)$ in (3) and collect the terms involving like powers of $x - a$. If we then equate the coefficients of the different powers of $x - a$ to zero we get:

$$g_0 f_0(0) = 0 ,$$
$$g_1 f_0(1) + g_0 f_1(0) = 0 ,$$
$$g_2 f_0(2) + g_1 f_1(1) + g_0 f_2(0) = 0 ,$$
$$g_3 f_0(3) + g_2 f_1(2) + g_1 f_2(1) + g_0 f_3(0) = 0 ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

the first two of these equations are satisfied by any values of g_0 and g_1 since $f_0(0) = f_0(1) = f_1(0) = 0$. By means of the remaining equations we can compute in succession $g_2, g_3, \ldots$, the general formula being:

$$g_\nu = \frac{-1}{\nu(\nu-1)}[g_{\nu-1} f_1(\nu-1) + g_{\nu-2} f_2(\nu-2) + \ldots + g_0 f_\nu(0)].$$

It remains to prove that the series (2) in which the coefficients have been thus determined converges when $|x - a| < R$. We will suppose that x is any quantity satisfying this inequality and we will take x' as a second quantity satisfying the relation

$$|x-a| < |x'-a| < R .$$

For the sake of brevity we will let $|x' - a| = K$.

Consider the developments:

$$(x'-a)p(x') = p_0(x'-a) + p_1(x'-a)^2 + \ldots ,$$
$$(x'-a)^2 q(x') = q_0(x'-a)^2 + q_1(x'-a)^3 + \ldots .$$

Since these series are both absolutely convergent the absolute values of the terms will all be smaller than some positive constant M:

$$|p_{\mu-1}| K^\mu < M , \quad |q_{\mu-2}| K^\mu < M .$$

We thus get:

$$|f_\mu(\rho)| \leqq |\rho| |p_{\mu-1}| + |q_{\mu-2}| < MK^{-\mu}(|\rho| + 1),$$

$$|g_\nu| < \frac{M}{\nu(\nu-1)}[|g_{\nu-1}| K^{-1}\nu + |g_{\nu-2}| K^{-2}(\nu-1) + \ldots + |g_0| K^{-\nu}].$$

Let us call the second member of this inequality c_ν. If we can prove that the series $\Sigma c_\nu |x-a|^\nu$ is convergent our theorem is established. Now we have:

$$c_{\nu+1} = \frac{M}{(\nu+1)\nu}[|g_\nu| K^{-1}(\nu+1) + |g_{\nu-1}| K^{-2}\nu + \ldots + |g_0| K^{-\nu-1}],$$

$$\frac{c_{\nu+1}}{c_\nu} = \frac{\nu-1}{\nu+1}K^{-1} + \frac{MK^{-1}}{\nu}\frac{|g_\nu|}{c_\nu},$$

from which, remembering that $|g_\nu| < c_\nu$, it follows that:

$$\lim_{\nu=\infty}\left[\frac{c_{\nu+1}}{c_\nu}|x-a|\right] = \frac{|x-a|}{K} < 1,$$

and the series $\Sigma c_\nu |x-a|^\nu$ converges.

The proof just given is essentially an application to the case of a non-singular point of the more general proof given by Frobenius in Crelle, Vol. 76. It may be noticed, however, that as here given the proof applies to the case in which p and q are real analytic functions of the real variable x without any reference whatever to imaginary values.

§ 2. *Equations with Analytic Coefficients have none but Analytic Solutions.*

It follows at once from the algorithm of the last section that the differential equation can have no solution, other than those there discussed, which is analytic at a.* This does not, however, prove that the solutions obtained in the last section are the only analytic solutions; there might, for instance, be other analytic solutions with a singular point at a. In order to consider this question we will introduce the idea of the analytic continuation of the solutions obtained in § 1. We thus easily deduce the theorem (cf. Heffter's Einleitung in die Theorie der linearen Differentialgleichungen, § 39): *A solution of equation* (1) *that is analytic in any part of a connected region throughout which* $p(x)$ *and* $q(x)$ *are analytic, is itself analytic throughout this region.* Thus the solutions above discussed include all analytic solutions of (1).†

* Cf. foot-note, p. 45.

† More accurately they include all solutions analytic at any part of the connected region including a throughout which both p and q are analytic. The region throughout which p and q are both analytic may, however, consist of two or more separate pieces in which case solutions of the equation analytic in one piece would not in general exist in the others. In such a case, however, we practically have a number of distinct differential equations.

Let us, however, look at the case in which p and q are real functions of the real variable x. Then, even though p and q are analytic, there is no reason *a priori* why the differential equation should not have, besides the analytic solutions obtained in the last section, other solutions which are not analytic. It is therefore of fundamental importance to establish the following theorem :

If throughout an interval AB of the x-axis the coefficients p and q of equation (1) *are real analytic functions of the real variable x, a real function y which at every point of AB satisfies equation* (1) *will be analytic throughout AB.*

It should be noticed that when we require that the function y should satisfy the differential equation at every point of AB we thereby require that it should have a first and a second derivative at every such point, and therefore, in particular, that y and its derivative y' should be continuous throughout AB.

In order to prove the theorem just stated let a be any point of the interval AB. Call the values which y and y' have at a α and β respectively. Consider now the analytic solution whose existence was established in § 1 :

$$\bar{y} = \alpha + \beta (x - a) + g_2 (x - a)^2 + \ldots$$

$y - \bar{y}$ is then a solution of the differential equation which together with its first derivative vanishes when $x = a$. Our proof will be complete if we can prove that $y - \bar{y}$ vanishes at every point of AB for we should then have $y = \bar{y} =$ an analytic solution.

It remains then to prove a theorem which can be proved with the same ease in the following more general form in which we do not require that p and q be analytic, but merely that they be continuous :

If throughout an interval AB p and q are continuous real functions of the real variable x, a real function y which satisfies equation (1) *at every point of AB and which together with its first derivative vanishes at a point a of the interval AB will vanish throughout AB.*

We will suppose that $p(x)$ and $q(x)$ are continuous at A and B as well as between these points.* It will then be possible to find a positive quantity M greater than the greatest numerical value of $p(x)$ and $q(x)$ in AB. We will suppose M to be also taken greater than $\frac{1}{4}$:

$$M > |p(x)|, \quad M > |q(x)|, \quad M > \tfrac{1}{4}.$$

We will consider first not the whole interval but only the interval from

* If this were not the case we should merely have to take two other points A' and B' between A and B and arbitrarily near to A and B respectively, and then to prove the theorem for the interval $A'B'$.

$a - 1/4M$ to $a + 1/4M$ or so much of this interval as lies between A and B. We assume that $y = 0$ and $y' = 0$ when $x = a$ and we wish to prove that $y = 0$ for all values of x between $a - 1/4M$ and $a + 1/4M$.

Consider the maxima of the functions $|y|$ and $|y'|$ in the interval just mentioned, and call the larger of these two quantities c, so that when

$$a - 1/4M \leqq x \leqq a + 1/4M \qquad |y| \leqq c \text{ and } |y'| \leqq c.$$

It will then be sufficient if we can prove that $c = 0$. It follows from the differential equation that for all points of the interval we are considering:

$$|y''| \leqq |p(x)|\,|y'| + |q(x)|\,|y| \leqq 2cM.$$

Now we see by applying the law of the mean* to the function y' that at any point x of our interval $(y' - 0)/(x - a)$ is equal to the value of y'' at some point between a and x. Accordingly:

$$|y'| \leqq 2Mc\,|x - a| \leqq c/2 \text{ (since } |x - a| \leqq 1/4M).$$

Applying the law of the mean to the function y we see that $(y - 0)/(x - a)$ is equal to the value of y' at some point between a and x, so that:

$$|y| \leqq \frac{c}{2}\,|x - a| \leqq \frac{c}{2} \cdot \frac{1}{4M} \leqq \frac{c}{2} \text{ (since } M > \tfrac{1}{4}).$$

We have thus shown that neither $|y|$ nor $|y'|$ exceeds the value $c/2$ at any point of our interval, while by hypothesis either $|y|$ or $|y'|$ has the value c at some point of this interval. It follows that $c \leqq c/2$, so that, since c cannot be negative, $c = 0$.

Having thus proved that y and y' vanish throughout the interval from $a - 1/4M$ to $a + 1/4M$ † we now proceed to extend our result to the whole interval AB. Since y is a solution of (1) which together with its first derivative vanishes when $x = a + 1/4M$, the reasoning just used shows that it must vanish throughout the interval from $a + 1/4M$ to $a + 2/4M$. Applying the same reasoning again we find that it vanishes from $a + 2/4M$ to $a + 3/4M$, etc. On the other hand, since y and y' vanish when $x = a - 1/4M$ we see that y vanishes from $a - 1/4M$ to $a - 2/4M$, from $a - 2/4M$ to $a - 3/4M$, etc. It follows that y vanishes throughout AB.

The proof just given is an application to the case of linear differential

* I. e. the theorem that if $f(x)$ is a continuous function and has a derivative: $(f(x) - f(a))/(x - a) = f'(\xi)$ where ξ is some point between a and x.

† Here and in the next few lines only so much of the intervals is meant as is included in AB.

equations of the proof given by Jordan in the revised edition of his Cours d'Analyse, Vol. III, p. 93.*

§ 3. *Equations with Real Coefficients. Existence Theorem.*

The last theorem established applies to equations in which p and q are any real continuous functions of the real variable x. In this section we will establish the following existence theorem for such equations analogous to the one established in § 1 for equations with analytic coefficients.

If in the interval AB p and q are continuous real functions of the real variable x, a real function y exists which satisfies equation (1) *at every point of AB and which has at the arbitrarily chosen point a of AB the arbitrary value α while its derivative has at this point the arbitrary value β.*

We will establish this theorem by the method of successive approximations first used for this purpose by Peano in 1887.† As a first approximation y_0 we will take the simplest function which satisfies the initial conditions $y_0(a) = \alpha$, $y_0'(a) = \beta$, i. e. the linear function :

$$y_0 = \alpha + \beta(x - a) .$$

We then compute a second approximation y_1 from the relation :

$$y_1'' + p(x)\, y_0' + q(x)\, y_0 = 0 ,$$

and the initial conditions $y_1(a) = \alpha$, $y_1'(a) = \beta$. This gives when we remember that $y_0' = \beta$:

$$y_1' - y_0' = -\int_a^x [\, p(x) y_0' + q(x) y_0]\, dx ,$$

and integrating again :

$$y_1 - y_0 = \int_a^x (y_1' - y_0')\, dx .$$

Proceeding in this way we compute each approximation y_n from the preceding approximation y_{n-1} by the relation :

$$y_n'' + p(x)\, y_{n-1}' + q(x)\, y_{n-1} = 0$$

and the initial conditions :

$$y_n(a) = \alpha , \quad y_n'(a) = \beta .$$

* The form in which Jordan gives this proof is far from satisfactory, but his proof may readily be made rigorous. Cf. also Lindelöf : Journal de Mathématique, 1894, p. 118.

† Cf. Math. Ann. Bd. 32. For other references see the *Bulletin*, loc. cit.

This gives :

$$y_n' - y_0' = -\int_a^x [p(x)\, y_{n-1}' + q(x)\, y_{n-1}]\, dx\,,$$

$$y_n - y_0 = \int_a^x (y_n' - y_0')\, dx\,.$$

We will now prove : 1) that as n increases indefinitely y_n approaches a definite limit y; 2) that this limit y satisfies the desired initial conditions; 3) that y satisfies the differential equation. Our theorem will then be established.

The problem will have a more familiar form if y_n is regarded as the sum of the first $n+1$ terms of a series. This can be done by letting $Y_1 = y_1 - y_0$, $Y_2 = y_2 - y_1, \ldots, Y_n = y_n - y_{n-1} \ldots$. Then y_n is the sum of the first $n+1$ terms of the series

$$y_0 + Y_1 + Y_2 + Y_3 + \ldots, \tag{4}$$

and our first problem is to prove that this series converges at all points of the interval AB. It is easy to so arrange our work as to prove at the same time the convergence of the second series :

$$y_0' + Y_1' + Y_2' + Y_3' + \ldots. \tag{5}$$

The functions $Y_1', Y_1, \ldots, Y_n', Y_n$ are computed by the formulæ :

$$Y_1' = -\int_a^x [p(x)\, y_0' + q(x)\, y_0]\, dx\,,$$

$$Y_1 = \int_a^x Y_1' dx\,,$$

$$Y_n' = -\int_a^x [p(x)\, Y_{n-1}' + q(x)\, Y_{n-1}]\, dx\,,$$

$$Y_n = \int_a^x Y_n' dx\,.$$

Let C be a positive quantity satisfying the inequalities $|y_0| < C$ $|y_0'| < C$ throughout AB.

Let $t = |x - x_0|$ and let l be a positive constant greater than 1 and satisfying for all points of AB the inequality $t < l$.

Let M be a positive constant satisfying throughout AB the inequalities $|p(x)| < M$, $|q(x)| < M$.

We have then evidently :

$$|Y_1'| \leqq \int_0^t 2CMdt = 2CMt\,,$$

$$|Y_1| \leqq \int_0^t 2CMtdt = \frac{2CMt^2}{2}\,.$$

The two expressions which we have here found, one greater than $|Y_1'|$ and the other greater than $|Y_1|$ may clearly be replaced by one and the same quantity $2CMlt$, which is greater than either of them. We thus get

$$|Y_1'| < 2CMlt\,, \quad |Y_1| < 2CMlt\,.$$

Proceeding in the same way :

$$|Y_2'| \leqq \int_0^t 2^2CM^2ltdt = \frac{2^2CM^2lt^2}{2\,!}\,,$$

$$|Y_2| \leqq \int_0^t \frac{2^2CM^2lt^2}{2!}\,dt = \frac{2^2CM^2lt^3}{3!}\,.$$

The two expressions on the right hand side may here again be replaced by a single one :

$$|Y_2'| < \frac{2^2CM^2l^2t^2}{2\,!}\,, \quad |Y_2| < \frac{2^2CM^2l^2t^2}{2!}\,.$$

At the next step we get :

$$|Y_3'| < \frac{2^3CM^3l^3t^3}{3\,!}\,, \quad |Y_3| < \frac{2^3CM^3l^3t^3}{3!}\,,$$

etc. It is clear then that the absolute values of the terms of the series (4) and (5) are less than the corresponding terms of the series :

$$C + C\,.\,2Mlt + C\,\frac{(2Mlt)^2}{2!} + C\,\frac{(2Mlt)^3}{3!} + \ldots,$$

and this being a convergent series of positive terms the series (4) and (5) are absolutely convergent throughout the interval AB. Moreover the terms of the series last written being smaller than the corresponding terms of the series :

$$C + C\,.\,2Ml^2 + C\,\frac{(2Ml^2)^2}{2!} + C\,\frac{(2Ml^2)^3}{3!} + \ldots,$$

and this being a convergent series of constant terms (i. e. terms independent

of t) it follows at once that the series (4) and (5) are uniformly convergent throughout AB. Series (5) represents therefore the derivative of (4).

It follows now at once that y satisfies the initial conditions we wish it to satisfy; for we have:

$$y = \lim_{n=\infty} y_n \text{ and } y' = \lim_{n=\infty} y'_n$$

and $y_n(a) = \alpha$, $y'_n(a) = \beta$, so that $y(a) = \alpha$, $y'(a) = \beta$.

It remains then merely to prove that y satisfies the differential equation at every point of AB. Now we have:

$$y'_n - y'_0 = -\int_a^x [p(x) y'_{n-1} + q(x) y_{n-1}] \, dx .$$

Let us here take the limit of each side as n becomes infinite, remembering that since y_{n-1} and y'_{n-1} approach their limits uniformly we have a right to take the limit under the sign of integration:

$$y' - y'_0 = -\int_a^x [p(x) y' + q(x) y] \, dx .$$

When we differentiate this equation we get:

$$y'' = -p(x) y' - q(x) y ,$$

i. e. y satisfies the differential equation at every point of AB.

By the method used in § 2 it follows at once that no other real functions y than those just obtained exist which satisfy equation (1) at every point of AB.

Finally we may note that the exponential series obtained in the above proof of convergence may be conveniently used, if we wish to employ the method here explained for numerical computation, to determine how large n must be taken in order that y_n should be a sufficiently close approximation for the purpose in hand.

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ON THE TRIPLE FOCUS OF A CARTESIAN.

By Dr. Carl C. Engberg, Lincoln, Neb.

In his treatise on "Higher Plane Curves," Salmon says: "If I and J be each of them a cusp, then the tangent at I or J counts three times among the I or J tangents; and there are from each point $n - g - 3$ other tangents. The $(n - g)^2$ foci are then made up of one which counts as nine, of $(n - g - 3) + (n - g - 3)$ which each counts as three, and $(n - g - 3)^2$ single foci. Of these last $(n - g - 3)$ are real, and the only other real focus is the intersection of the tangents at I and J, which is commonly called a triple focus as counting for three among the real foci, though if we took into account imaginary as well as real foci, it ought to be regarded as a 9-tuple focus."

According to this theory, the foci of a Cartesian oval, the class of which is six, would be as follows: one 9-tuple focus, six triple, and nine single foci. Of these three single foci are real, and the only other real focus is the intersection of the cuspo-tangents, or, as it is commonly called, the triple focus.

This is on the condition that the line IJ does not count among the tangents to the curve. If IJ counts once among the tangents from I or J to the Cartesian, we get two real foci in finite regions beside the real triple focus, and also a real focus at infinity. Here a difficulty presents itself. It is a well known fact that a Cartesian Oval has three foci on its axis. In addition it has, in a certain sense, a fourth focus on the axis, namely, the point on the line which is infinitely distant. This we may show as follows:

From the circular points I, J at infinity four tangents may be drawn to a nodal bicircular quartic. Now we know that, if from a point on a curve, four tangents can be drawn to the curve, the anharmonic ratio of the four sets of tangents thus formed is constant; whence the tangents intersect in sixteen points, four points on each of four circles, which intersect each other orthogonally. Four of these foci, the intersections of conjugate lines, are real, the other twelve are imaginary. Now the inverse of a nodal bicircular quartic with respect to a focus is a Cartesian. Therefore, inverting with respect to one of the real foci, the circle containing these foci inverts into a straight line, three of the foci invert into the finite foci of the Cartesian, while the fourth focus, the centre of inversion, inverts into the infinite point on the line.

If the foci of a Cartesian follow the law given by Salmon, we shall have either three real single finite foci and one real triple focus, but no focus at infinity; or we shall have two real single finite foci, one real infinite, and one real triple focus. The first of these cases evidently does not hold. If now we

can show that a Cartesian has three real finite foci in addition to the triple focus, we have shown that the second case does not hold. This we may show as follows:

The equation of a Cartesian can readily be put into the form

$$s^2 - k^3(x - 1),$$

where $s = 0$ is the equation of a circle. This form of the equation shows that the I, J points are cusps, and that the cuspidal tangents intersect in the centre of s, which is therefore the triple focus, or cuspo-focus of the Cartesian.

Writing ρ^2 for $x^2 + y^2$, we have for the equation of a complete Cartesian, a focus being taken for pole,

$$(\rho^2 - 2Bx + C^2)^2 - 4A^2\rho^2 = 0. \tag{1}$$

The equivalent form

$$\{(x - B)^2 + y^2 - B^2 - 2A^2 + C^2\}^2 - 4A^2(A^2 - C^2 + 2Bx) = 0$$

shows that $(x = B, y = 0)$ are the co-ordinates of the cuspo-focus.

We proceed to identify (1) with the vectorial form

$$m\rho' - \rho = K$$

that is, with

$$m(\rho^2 - 2cx + c^2)^{\frac{1}{2}} - \rho = K,$$

or

$$\left\{\rho^2 - \frac{2m^2cx}{m^2 - 1} + \frac{m^2c^2 - K^2}{m^2 - 1}\right\}^2 - \frac{4K^2\rho^2}{(m^2 - 1)^2}. \tag{2}$$

The equalities

$$A^2 = \frac{K^2}{(m^2 - 1)^2}, \quad B^2 = \frac{m^2c}{m^2 - 1}, \quad C^2 = \frac{m^2c^2 - K^2}{m^2 - 1}$$

give

$$c^2 - (B^2 + C^2 - A^2)\frac{c}{B} + C^2 = 0, \tag{3}$$

the roots of (3) representing, of course, the two axial foci not at the origin. The roots are

$$\frac{(B^2 + C^2 - A^2) \pm (A^4 + B^4 + C^4 - 2B^2C^2 - 2C^2A^2 - 2A^2B^2)^{\frac{1}{2}}}{2B},$$

neither of which, in a true Cartesian, reduces to B.

Now returning to the tangents, the term "triple tangent" may be interpreted in two ways: 1. Three of the tangents to the curve have become coincident at the cusp; 2. The tangent at the cusp really contains only two of the

tangents to the curve, but is called triple tangent because it touches the curve in three coincident points. As has been shown above, the first interpretation does not account for all the real foci of the Cartesian. We are thus forced to the second interpretation. This, however, shows that the foci of a Cartesian do not follow Plücker's law; and gives the following distribution of I or J tangents: Two coincident cuspidal tangents, the line IJ, and three other distinct tangents. According to this our foci are as follows: One which counts for four, eight double, and sixteen single foci. Of these foci, four single are real, three of them finite and one infinite, and the only other real focus is the quadruple one, which should be called double as counting for two among the real foci. Thus the focus which is generally considered a triple focus is really only a double focus. The single foci correspond to the sixteen foci of the bicircular quartic. As their properties and distribution are well known, nothing further need to be said about them.

Summing up these results, we see that the term "triple tangent" does not apply to the Cartesian oval, in connection with the focal properties, in the sense in which it is used in the general theory of foci, there being four tangents from each cusp in addition to the cuspidal tangents, while the class of the Cartesian is only 6. In view of this it seems to me more fitting to use the terms "cuspo-tangent" and "cuspo-focus," since these terms are not misleading as are the terms triple tangent and triple focus.

DIRECT DERIVATION OF THE ORDINARY CANONICAL SYSTEM OF ELLIPTIC ELEMENTS EMPLOYED IN THE PROBLEM OF THREE BODIES.

By Prof. Ormond Stone, Charlottesville, Va.

The canonical equations of motion may be written

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{1}$$

in which

$$H = \Sigma p_i \frac{dq_i}{dt} - T - F \tag{2}$$

is a constant, and

$$p_i = \frac{\partial T}{\partial \frac{dq_i}{dt}}. \tag{3}$$

T is a function of the q_i's and $\frac{dq_i}{dt}$'s but H is expressed as a function of the p_i's and q_i's.

We will consider the problem of three bodies in which

$$T = \tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2} r^2 \left[\frac{dw}{dt}\right]^2, \tag{4}$$

$$F = \frac{\mu}{r} + R_1, \tag{5}$$

$$H = \tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2} r^2 \left[\frac{dw}{dt}\right]^2 - \frac{\mu}{r} - R_1, \tag{6}$$

where r, w are the radius vector and longitude in orbit of the disturbed body, R_1 is a function of the coordinates of both the disturbed and disturbing bodies, and μ is a constant.

As q_i's we will select

$q_1 =$ the mean anomaly,

$q_2 =$ the angular distance of the perihelion from the node,

$q_3 =$ the longitude of the node counted from a fixed point in the fundamental plane; in each case reference being had to the instantaneous Keplerian ellipse. It remains to find p_1, p_2, p_3.

dq_1/dt is independent of dq_2/dt and dq_3/dt; it is also independent of the form of the orbit. In deriving p_1, therefore, by means of (3), we may write T_0 in the place of T, the former being obtained on the assumption that the orbit

is a circle whose radius is the semi-major axis, a, of the instantaneous ellipse. In the instantaneous ellipse, instead of (6) we have

$$\tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2}r^2\left[\frac{dw}{dt}\right]^2 - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,; \tag{7}$$

In the corresponding circle $dr/dt = 0$, $r = a$, $dw/dt = dq_1/dt$; whence

$$T_0 = \tfrac{1}{2}a^2\left[\frac{dq_1}{dt}\right]^2, \tag{8}$$

and

$$p_1 = \frac{\partial T_0}{\partial \frac{dq_1}{dt}} = a^2\frac{dq_1}{dt}. \tag{9}$$

Also, from (7), on the same assumption,

$$\tfrac{1}{2}a^2\left[\frac{dq_1}{dt}\right]^2 - \frac{\mu}{2a} = 0\,, \tag{10}$$

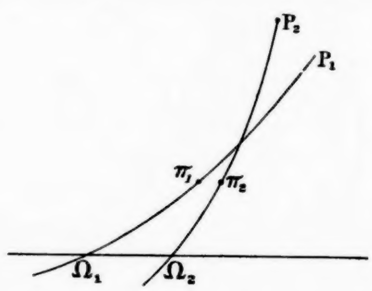
or

$$\frac{dq_1}{dt} = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}. \tag{11}$$

Combining (9) and (11), we have

$$p_1 = \sqrt{\mu a}\,. \tag{a}$$

A reference to the accompanying figure (in which P_1, P_2 are successive positions of the disturbed body on the celestial sphere, and Ω_1, Ω_2; π_1, π_2 are



the corresponding positions of the instantaneous node and perihelion) will show that

$$\frac{dw}{dt} = \frac{dv}{dt} + \frac{dq_2}{dt} + \frac{dq_3}{dt}\cos i\,, \tag{12}$$

where v is the instantaneous true anomaly, and i is the inclination of the

instantaneous orbit to the fundamental plane; and since dq_2/dt is independent of dr/dt, dv/dt, and dq_3/dt, equations (3), (4), and (12) give

$$p_2 = \frac{\partial T}{\partial \frac{dq_2}{dt}} = r^2 \frac{dw}{dt} \,. \tag{13}$$

Substituting in (7),

$$\tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2}\frac{p_2^2}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,. \tag{14}$$

This equation holds true for all points of the instantaneous ellipse. For the maximum and minimum values of r, since for these values $dr/dt = 0$, we have

$$\tfrac{1}{2}\frac{p_2^2}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,,$$

or

$$r^2 - 2ar + \frac{a}{\mu}p_2^2 = 0\,. \tag{15}$$

Calling the roots of this equation $r_1 = a(1 - e)$ and $r_2 = a(1 + e)$, as given by the properties of the ellipse, the last term of (15) gives

$$r_1 r_2 = \frac{a}{\mu}p_2^2 = a^2(1 - e^2)\,,$$

or

$$p_2 = \sqrt{\mu a(1 - e^2)} = p_1\sqrt{1 - e^2}\,, \tag{b}$$

in which, of course, e is the eccentricity of the instantaneous ellipse.

Finally, since dq_3/dt is independent of dr/dt, dv/dt, dq_2/dt, equations (3), (4), and (12) give

$$p_3 = \frac{\partial T}{\partial \frac{dq_3}{dt}} = r^2\frac{dw}{dt}\cos i = p_2 \cos i\,. \tag{c}$$

A comparison of (7) with (6) gives

$$H = -\frac{\mu}{2a} - R_1 = -\frac{\mu^2}{2p_1^2} - R_1 = -R\,,$$

in which R is the expression used by Delaunay in his Théorie de la Lune.

The derivation of the ordinary canonical system here given has the following apparent advantages: 1. The use of Hamilton's principal function is avoided; 2. The argument forming Chap. 36 of Jacobi's Vorlesungen über Dynamik, or that forming Vol. I, Art. 59, of Tisserand's Mécanique Céleste, is rendered unnecessary in applying the system to the theory of perturbations; 3. The reason for the addition of the term $\mu^2/(2p_1^2)$ to the perturbative function is shown, without a special investigation, such as that given in Vol. I, Art. 5, of Delaunay's Théorie de la Lune.

A BINOMIAL THEOREM, EXPRESSED IN FORM OF A FACTORIAL, WHICH IS *ALWAYS* CONVERGENT.*

By Chas. H. Kummell, Washington, D. C.

The binomial series:

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n}{1}\cdot\frac{n-1}{2}x^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3 + \ldots$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-m+1}{m}x^m + \ldots$$

$$= 1 + \frac{n}{1}x\left(1 + \frac{n-1}{2}x\left(1 + \frac{n-2}{3}\left(1 + \ldots\right.\right.\right. \quad (1)$$

is true for any value of x only if n is a positive integer. In all other cases we must have

$$1 > x > -1.$$

About six years ago Dr. Martin communicated to the Mathematical Section of the Philosophical Society a remarkable series for $\sqrt{a^2+b}$, which had been discovered by C. A. Roberts, as follows:

$$\sqrt{a^2+b} = a + \frac{b}{2a}\left(1 - \frac{1}{q_1}\left(1 + \frac{1}{q_2}\left(1 + \frac{1}{q_3}\left(1 + \ldots\right.\right.\right.\right. \quad (2)$$

where

$$q_1 = \frac{4a^2}{b} + 2\,;\ q_2 = q_1^2 - 2\,;\ q_3 = q_2^2 - 2 \ldots. \quad (3)$$

Since the q's increase to infinity, it is evident that this series converges for any value of a and b as I proved in my article "On the method of continued identity," Annals of Mathematics, Vol. 5, page 3, page 92. (This was an attempt to apply this form to the development of any function, which, however, was not practically successful except in yielding a remarkably convenient method of solving higher numerical equations.)

After that I published in the Mathematical Magazine "A new and expeditious method of computing the square root," which already contained the essential features of the general method though somewhat hidden. It is as follows:

We have identically

$$\sqrt{a^2+b} = \frac{b}{2a}\sqrt{\frac{4a^2}{b^2}(a^2+b)} = \frac{b}{2a}\sqrt{\tfrac{1}{4}q_1^2 - 1} \text{ if } q_1 = \frac{4a^2}{b} + 2$$

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* Read before the Philosophical Society, Washington, D. C.

similarly

$$\sqrt{\tfrac{1}{4} q_1^2 - 1} = \frac{1}{q_1} \sqrt{q_1^2 (\tfrac{1}{4} q_1^2 - 1)} = \frac{1}{q_1} \sqrt{\tfrac{1}{4} q_2^2 - 1} \text{ if } q_2 = q_1^2 - 2$$

$$\sqrt{\tfrac{1}{4} q_2^2 - 1} = \frac{1}{q_2} \sqrt{q_2^2 (\tfrac{1}{4} q_2^2 - 1)} = \frac{1}{q_2} \sqrt{\tfrac{1}{4} q_3^2 - 1} \text{ if } q_3 = q_2^2 - 2\,.$$

We have then

$$\begin{aligned}\sqrt{a^2 + b} &= \frac{b}{2a} \sqrt{\tfrac{1}{4} q_1^2 - 1} = \frac{b}{2a} \cdot \frac{q_1}{2} \text{ approx.} \\ &= \frac{b}{2a} \cdot \frac{1}{q_1} \sqrt{\tfrac{1}{4} q_2^2 - 1} = \frac{b}{2a} \cdot \frac{1}{q_1} \cdot \frac{q_2}{2} \text{ approx.} \\ &= \frac{b}{2a} \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} \sqrt{\tfrac{1}{4} q_3^2 - 1} = \frac{b}{2a} \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} \cdot \frac{q_3}{2} \text{ approx.}\end{aligned} \tag{4}$$

etc.

These approximate values, since they are expressed by the identical auxiliaries q as in Robert's series, are likewise convergent for any a and b. They are indeed the sums of the terms of that series since

$$\begin{aligned}a + \frac{b}{2a} &= \frac{b}{2a} \cdot \frac{q_1}{2} \\ a + \frac{b}{2a} \left[1 - \frac{1}{q_1} \right] &= \frac{b}{2a} \cdot \frac{1}{q_1} \cdot \frac{q_2}{2} \\ a + \frac{b}{2a} \left[1 - \frac{1}{q_1} \left[1 + \frac{1}{q_2} \right] \right] &= \frac{b}{2a} \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} \cdot \frac{q_3}{2}\,,\end{aligned} \tag{5}$$

etc.

The generalization of this method for the nth root did not occur to me because then the successive remainders could not be $= 1$, and I considered this erroneously essential. The following, however, is the correct mode of generalization :

We have

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \left[(na^{n-1})^n (a^n \pm b) \right]^{\frac{1}{n}} = \frac{1}{na^{n-1}} (a_1^n - b_1)^{\frac{1}{n}} \tag{6_1}$$

if

$$a_1 = na^n \pm b\,;\ b_1 = a_1^n - (na^{n-1})^n (a^n \pm b) \tag{7_1}$$

$$(a_1^n - b_1)^{\frac{1}{n}} = \frac{1}{na_1^{n-1}} \left[(na_1^{n-1})^n (a_1^n - b_1) \right]^{\frac{1}{n}} = \frac{1}{na_1^{n-1}} (a_2^n - b_2)^{\frac{1}{n}} \tag{6_2}$$

if

$$a_2 = na_1^n - b_1\,;\ b_2 = a_2^n - (na_1^{n-1})^n (a_1^n - b_1)\,, \tag{7_2}$$

etc., and we have

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \cdot \frac{1}{na_1^{n-1}} \cdot \frac{1}{na_2^{n-1}} \cdots \frac{1}{na_{m-1}^{n-1}} (a_m^n - b_m)^{\frac{1}{n}}\,. \tag{6}$$

Now whatever the ratio between a^n and b, even if $b > a^n$ it follows from (7)

$$a_1{}^n > b_1\,;\ a_2{}^n > b_2,\ \text{etc.},$$

and the binomial series can be applied without restriction to any form of (6), and if the remainders $b_1, b_2, \ldots$ are small enough with respect to $a_1{}^n, a_2{}^n, \ldots$ to be neglected, we have the approximate values :

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \cdot a_1, \text{ approximately,}$$

$$= \frac{1}{na^{n-1}} \cdot \frac{1}{na_1{}^{n-1}} \cdot a_2, \text{ approximately,} \tag{8}$$

etc.

In order to prove that these approximate values converged to the true value we give to the factorial (6) another form. Let

$$\frac{b}{a^n} = x\,;\ \frac{b_1}{a_1{}^n} = x_1\,;\ \frac{b_2}{a_2{}^n} = x_2,\ \text{etc.} \tag{9}$$

Then we have :

$$(a^n \pm b)^{\frac{1}{n}} = a\,(1 \pm x)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \cdot a_1\,(1 - x_1)^{\frac{1}{n}} = a\left[1 \pm \frac{x}{n}\right](1 - x_1)^{\frac{1}{n}} \tag{10_1}$$

$$= a\left[1 \pm \frac{x}{n}\right]\left[1 - \frac{x_1}{n}\right](1 - x_2)^{\frac{1}{n}} \tag{10_2}$$

. . . .

$$= a\left[1 \pm \frac{x}{n}\right]\left[1 - \frac{x_1}{n}\right]\left[1 - \frac{x_2}{n}\right]\cdots\left[1 - \frac{x_{m-1}}{n}\right](1 - x_m)^{\frac{1}{n}}, \tag{10_m}$$

and the scale of relation becomes :

$$x_1 = 1 - \left[1 \pm \frac{x}{n}\right]^{-n}(1 \pm x), \tag{11_1}$$

$$x_2 = 1 - \left[1 - \frac{x_1}{n}\right]^{-n}(1 - x_1), \tag{11_2}$$

. . . .

$$x_m = 1 - \left[1 - \frac{x_{m-1}}{n}\right]^{-n}(1 - x_{m-1}). \tag{11_m}$$

Since even if $x > 1$ we have necessarily $x_1\,;\ x_2, \ldots, x_m < 1$, we may apply

the binomial series to the second terms, except the first, if $\frac{x}{n} > 1$. We have then

$$x_m = 1 - \left[1 + x_{m-1} + \frac{n+1}{2n} x_{m-1}^2 + \frac{n+1}{2n} \cdot \frac{n+2}{3n} x_{m-1}^3 + \ldots\right](1 - x_{m-1})$$

$$= \left\{1 - \frac{n+1}{2n}\right\} x_{m-1}^2 + \frac{n+1}{2n}\left\{1 - \frac{n+2}{3n}\right\} x_{m-1}^3 + \ldots$$

$$= \frac{n-1}{2n} x_{m-1}^2 + \frac{n+1}{n} \cdot \frac{n-1}{3n} x_{m-1}^3 + \ldots . \tag{12}$$

If, therefore, x_{m-1} is a small quantity of the first order then x_m is of the second order, and the smaller x_{m-1} the more nearly we can put

$$x_m = \frac{n-1}{2n} x_{m-1}^2 .$$

Since, then, the x diminish continually, therefore (10), and hence, also (8), are *always* convergent.

If n is not an integer, and therefore $\frac{1}{n}$ any proper or improper fraction, these conclusions are not affected and these factorials remain convergent though practically useless since then the scales of relation require extractions of high roots. The forms given are thus restricted in practice to the extraction of high roots, and if $N^{\frac{p}{q}}$ was required we place

$$N^{\frac{1}{q}} = (a^q \pm b)^{\frac{1}{q}} = a(1 \pm x)^{\frac{1}{q}}$$

and raise the result to the pth power.

In case $b > a^n$ or $x > 1$, the forms are however so slowly convergent that they are impracticable, while the binomial series is absurd.

Additional Note. Instead of deriving (10) from (6) it can be formed independently as follows:

We have $(1 + x)^{\frac{1}{n}} = 1 + \frac{x}{n}$, approximately, using two terms of the binomial series. Assume

$$(1 + x)^{\frac{1}{n}} = \left\{1 + \frac{x}{n}\right\}(1 - x_1)^{\frac{1}{n}}$$

then

$$x_1 = 1 - \left\{1 + \frac{x}{n}\right\}^{-n}(1 + x), \text{ the same as } (11_1).$$

In the same manner we can take

$$(1+x)^{\frac{1}{n}} = \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right](1 - x_1)^{\frac{1}{n}} \tag{13}$$

whence the scale of relation

$$x_1 = 1 - \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^{-n}(1+x)\,. \tag{14$_1$}$$

But we have

$$x_1 = 1 - \Big[1 - \frac{n}{1}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right] + \frac{n}{1} \cdot \frac{n+1}{2}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^2$$

$$- \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^3 + \ldots\Big](1+x)$$

$$= \left[\frac{n(n+1)}{1\,.\,2} \cdot \frac{n-1}{n^3} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3n^3} - \frac{n(n+1)}{1\,.\,2n^2}\right]x^3 + \ldots$$

$$= \frac{(n+1)(n-1)}{1\,.\,2\,.\,3}\,\frac{x^3}{n^2} + \ldots,$$

thus if x is small of the first order, x_1 is of the third, and the convergence of the factorial

$$(1+x)^{\frac{1}{n}} = \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]\left[1 - \frac{x_1}{n} - \frac{n-1}{2} \cdot \frac{x_1^2}{n^2}\right] \cdots$$

$$\left[1 - \frac{x_{m-1}}{n} - \frac{n-1}{2} \cdot \frac{x_{m-1}^2}{n^2}\right](1 - x_m)^{\frac{1}{n}}$$

is cubic.

By taking four terms of the binomial series for first factor we can form a factorial which has a quartic convergence and so on. However, since the scales of relation thus become more complicated, it is not apparent that there is any advantage in increasing the convergence in this manner. Moreover, if $x > 1$, then the first step becomes less advantageous the more terms are taken in the first factor.

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